Equation (4) of The BRDF page,

$$L_r(\theta_r, \phi_r) = \int_{2\pi_i} L_i(\theta_i, \phi_i)\, BRDF(\theta_i, \phi_i, \theta_r, \phi_r)\, \cos\theta_i \, d\Omega_i \, ,$$

shows how the BRDF is used in the radiative transfer equation (e.g., in HydroLight), which is always working with *radiances*. In Monte Carlo simulations, you are tracking many individual *rays* as they interact with the medium and its boundary surfaces. In this case, the BRDF must be used as a *probability distribution function* (PDF) to determine the direction and weight of the reflected ray whenever a ray hits the boundary surface. This is a tricky business, and the step-by-step process is as follows.

Computing the Reflected Ray Weight and Direction from a BRDF

Given: A ray with weight w_i is incident onto the surface in direction (θ_i, ϕ_i). The BRDF of the surface is known.

Needed: The weight w_r and direction (θ_r, ϕ_r) of the reflected ray.

Since the input direction (θ_i, ϕ_i) is known, $BRDF(\theta_i, \phi_i, \theta_r, \phi_r)$ can be viewed as an (unnormalized) bivariate PDF for the reflected angles θ_r and ϕ_r. Note that, in general, these angles are correlated. Proceed as follows:

1. Compute the directional-hemispherical reflectance for the given (θ_i, ϕ_i):

$$\begin{aligned} \rho^{\mathrm{dh}}(\theta_i, \phi_i) &= \& \iint_{2\pi_i} BRDF(\theta_i, \phi_i, \theta_r, \phi_r)\, \cos\theta_r \, d\Omega_r \\ &= \& \int_0^{2\pi} \int_0^{\pi/2} BRDF(\theta_i, \phi_i, \theta_r, \phi_r)\, \cos\theta_r \, \sin\theta_r \, d\theta_r \, d\phi_r \, . \end{aligned} \tag{1}$$

2. The reflected ray weight is then

$$w_r = \rho^{\mathrm{dh}}(\theta_i, \phi_i) w_i \, . \tag{2}$$

3. Compute the cumulative distribution function (CDF) for ϕ_r by

$$CDF_\phi(\phi_r) = \frac{1}{\rho^{\mathrm{dh}}(\theta_i, \phi_i)} \int_0^{\phi_r} \int_0^{\pi/2} BRDF(\theta_i, \phi_i, \theta, \phi)\, \cos\theta \, \sin\theta \, d\theta \, d\phi \, . \tag{3}$$

   Note that the directional-hemispherical reflectance is being used to convert the BRDF into a normalized bivariate PDF for θ_r and ϕ_r. We are then "integrating out" the θ_r dependence to leave a PDF for ϕ_r, which is then being used to construct the CDF for ϕ_r.

4. Draw a random number $\Re$ from a uniform [0,1] distribution. Solve the equation

$$\Re = CDF_\phi(\phi_r) \tag{4}$$

for ϕ_r. This is the randomly determined azimuthal angle of the reflected ray.

5. Compute the CDF for angle θ_r from

$$CDF_\theta(\theta_r) = \frac{\int_0^{\theta_r} BRDF(\theta_i, \phi_i, \theta, \phi_r) \cos\theta \sin\theta \, d\theta}{\int_0^{\pi/2} BRDF(\theta_i, \phi_i, \theta, \phi_r) \cos\theta \sin\theta \, d\theta} \ . \tag{5}$$

Note that the angle ϕ_r determined in step 4 is used in the BRDF in Eq. (likesection5) when evaluating the θ integrals. This is how the correlation between θ_r and ϕ_r is accounted for in the determination of the reflection angles.

6. Draw a new random number $\Re$ from a uniform [0,1] distribution and solve the equation

$$\Re = CDF_\theta(\theta_r) \tag{6}$$

for θ_r. This is the randomly determined polar angle of the reflected ray. You can now send the new ray on its way.

For all but the simplest BRDFs, Eqs. (likesection1) to (likesection6) all must be evaluated numerically for each ray, which can be an enormous computer cost when billions of rays are being traced.

A Simple Example

The Minnaert BRDF is

$$BRDF_{\text{Minn}}(\theta_i, \phi_i, \theta_r, \phi_r) = \frac{\rho}{\pi} (\cos\theta_i \cos\theta_r)^k \ . \tag{7}$$

[Comment: This BRDF was invented to explain the curious fact that the full moon appears almost uniformly bright from the center to the edge of the lunar disk. If the lunar dust were a Lambertian reflector, the full moon would appear bright at the center and darker at the edge. However, the Minnaert BRDF agrees with observation over only a limited range of angles.]

Note that for $k = 0$ this reduces to the Lambertian BRDF. Equations (likesection1) to (likesection6) can be evaluated analytically for the Minnaert BRDF. Equation (likesection1) evaluates to

$$\rho^{\text{dh}} = \frac{2\rho}{k+2} \cos^k\theta_i \ ,$$

which reduces to $\rho^{\text{dh}} = \rho$ for a Lambertian surface. Equation (likesection3) gives just

$$CDF_\phi(\phi_r) = \frac{\phi_r}{2\pi} \ .$$

Inserting this into Eq. (likesection4) and solving for ϕ_r gives

$$\phi_r = 2\pi\Re \ .$$

Thus the azimuthal angle is uniformly distributed over 2π radians. The CDF for θ_r as given by (likesection5) is

$$CDF_\theta(\theta_r) = 1 - \cos^{k+2}\theta_r \ .$$

Equation (likesection6) then gives

$$\theta_r = \cos^{-1}\left(\sqrt[k+2]{\Re}\right) \ ,$$

after noting that $1 - \Re$ has the same uniform distribution as $\Re$. For a Lambertian surface, the randomly generated θ_r angles are distributed as $\cos^{-1}(\sqrt{\Re})$, which certainly is not intuitive. However, this distribution is precisely what is necessary to make the number of reflected rays *per unit solid angle* proportional to $\cos\theta_r$, as required for a Lambertian surface. See the additional discussion of this on the Lambertian BRDFs page.