The radiative transfer equation is a statement of energy conservation in the sense that it accounts for all the losses and gains to a beam of light moving through the water along a path in a fixed direction. We now derive a useful conservation statement that holds at a fixed point in the water, through which light is moving in all directions.

The desired result is obtained by integrating the 1-D, time-independent, source-free RTE

$$\cos\theta\frac{dL(z,\theta,\phi,\lambda)}{dz} = \& - c(z,\lambda)\,L(z,\theta,\phi,\lambda)$$

$$+\& \int_0^{2\pi}\int_0^{\pi} L(z,\theta',\phi',\lambda)\,\beta(z,\theta',\phi'\to\theta,\phi,\lambda)\sin\theta'd\theta'd\phi' \tag{1}$$

over all directions. Dropping the wavelength argument for brevity and writing the differential element of solid angle $\sin\theta d\theta d\phi$ as $d\Omega(\theta,\phi)$, the left hand side of Eq. (1) yields

$$\int_0^{2\pi}\int_0^{\pi}\cos\theta\frac{dL(z,\theta,\phi)}{dz}d\Omega(\theta,\phi) = \&\frac{d}{dz}\int_0^{2\pi}\int_0^{\pi} L(z,\theta,\phi)\cos\theta d\Omega(\theta,\phi)$$

$$= \&\frac{d}{dz}[E_{\mathrm{d}}(z)-E_{\mathrm{u}}(z)] \tag{2}$$

after noting that $\cos\theta < 0$ for $\pi/2 < \theta \le \pi$ and recalling the definitions of the upwelling and downwelling plane irradiances as integrals of the radiance. The $-cL$ term becomes

$$\iint -c(z)L(z,\theta,\phi)d\Omega(\theta,\phi) = \& - c(z)\iint L(z,\theta,\phi)d\Omega(\theta,\phi)$$

$$= \& - c(z)E_{\mathrm{o}}(z)\ , \tag{3}$$

where the double integration over all directions is the same as shown in Eq. (2), and $E_{\mathrm{o}}(z)$ is the scalar irradiance. The elastic scatter path function gives

$$.\&\iint\left[\iint L(z,\theta',\phi')\,\beta(z,\theta',\phi'\to\theta,\phi)d\Omega(\theta',\phi')\right]d\Omega(\theta,\phi)$$

$$= \&\iint L(z,\theta',\phi')\left[\iint\beta(z,\theta',\phi'\to\theta,\phi)d\Omega(\theta,\phi)\right]d\Omega(\theta',\phi')$$

$$= \&b(z)\iint L(z,\theta',\phi')d\Omega(\theta',\phi')$$

$$= \&b(z)E_{\mathrm{o}}(z)\ . \tag{4}$$

Here we recall that the integral of the volume scattering function over all directions is the scattering coefficient.

Collecting terms (2)-(4) resulting from the directional integration of the RTE, we have

$$\frac{d}{dz}[E_{\mathrm{d}}-E_{\mathrm{u}}] = -cE_{\mathrm{o}}+bE_{\mathrm{o}}\ ,$$

or

$$\frac{d}{dz}\big[E_{\mathrm{d}}(z,\lambda) - E_{\mathrm{u}}(z,\lambda)\big] = -a(z,\lambda)E_{\mathrm{o}}(z,\lambda) \quad (\mathrm{W\,m^{-3}\,nm^{-1}})\,, \tag{5}$$

which is the desired result. This equation is known as Gershun's law (Gershun (1936) and Gershun (1939)).

The physical significance of Eq. (5) is that it relates the depth rate of change of the net irradiance $E_{\mathrm{d}} - E_{\mathrm{u}}$ to the absorption coefficient a and the scalar irradiance E_{o}. If inelastic scattering (fluorescence and Raman scattering) and internal sources (such as bioluminescence) are negligible at the wavelength of interest, then Eq. (5) can be used to obtain the absorption coefficient a from *in situ* measurements of the irradiance triplet $E_{\mathrm{d}}, E_{\mathrm{u}}$, and E_{o}. This is an example of an *explicit inverse model*—a model that retrieves an inherent optical property from measurements of the light field.

Voss (1989) used Gershun's law (5) to recover a values to within an estimated error of order 20%. Inelastic scattering and internal source effects were reasonably assumed to be negligible in his study. The needed irradiances were all computed from a measured radiance distribution, so that no intercalibration of instruments was required.

A more general development can be made to account for internal sources or inelastic scatter and for 3-D and time-dependent light fields, as shown in Light and Water section 5.10. The result is known as the *divergence law for irradiance*:

$$\frac{1}{v}\frac{\partial E_{\mathrm{o}}}{\partial t} + \nabla \cdot \vec{E} = -aE_{\mathrm{o}} + E_{\mathrm{o}}^{S}\,. \tag{6}$$

Here v is the speed of light in the water, $\vec{E}$ is the vector irradiance, and E_{o}^{S} is a source term. For time-independent, 1D, source-free water, this equation reduces to Eq. (5).

Maffione et al. (1993) determined absorption values by writing the source-free form of the 3-D divergence law in spherical coordinates and applying the result to irradiance measurements made using an underwater, artificial, *isotropic* light source. The artificial light source allowed measurements to be made at night, thus there was no inelastic scattering from other wavelengths. Their instrument did not require absolute radiometric calibration.

Note, however, Gershun's law will give *incorrect* absorption values if naively applied to waters and wavelengths where inelastic processes such as Raman scattering or fluorescence are significant. For this reason, and because of calibration difficulties if different instruments are used to measure $E_{\mathrm{d}}, E_{\mathrm{u}}$, and E_{o}, Gershun's law is seldom used as a way to measure absorption. Nevertheless, it is sometimes a useful check on the internal consistency of numerical models or measured data, and it leads to a convenient way of calculating radiant heating rates.

Heating Rates in the Upper Ocean

Gershun's law has a very important application in the computation of heating rates in the upper ocean. The rate of heating of water depends on how much scalar irradiance is available and on the total absorption coefficient of the water (pure water plus all other constituents). Combining the First Law of Thermodynamics (i.e., conservation of energy) with Eq. (5)

gives

$$\frac{\partial T}{\partial t}(z, \text{at } \lambda) = \frac{1}{c_v\rho}\, a(z,\lambda)E_\text{o}(z,\lambda) = -\frac{1}{c_v\rho}\frac{\partial[E_\text{d}(z,\lambda) - E_\text{u}(z,\lambda)]}{\partial z} \quad \left[\frac{\text{deg C}}{\text{sec}}\right].$$

Here

- T is the temperature in deg C

- t is the time in seconds

- $\frac{\partial T}{\partial t}(z, \text{at } \lambda)$ denotes the rate of change of temperature at depth z due to energy absorbed at wavelength λ. It does not mean that temperature is a function of wavelength.

- $c_v = 3900$ J (kg deg C)$^{-1}$ is the specific heat of sea water at constant volume

- $\rho = 1025$ kg m^{-3} is the density of sea water

This equation is usually applied with wavelength-integrated irradiances. In ocean modeling, "short-wave" radiation is usually taken to be the range of 400-1000 nm. Letting

$$E_\text{d,u}(z) = \int_{400}^{1000} E_\text{d,u}(z,\lambda)d\lambda$$

gives

$$\frac{\partial T}{\partial t}(z, \text{short-wave}) = -\frac{1}{c_v\rho}\frac{\partial[E_\text{d}(z) - E_\text{u}(z)]}{\partial z}. \tag{7}$$

In optically deep water, $E_\text{u}(z) << E_\text{d}(z)$, and the upwelling irradiance is usually dropped. The downwelling irradiance is then modeled, e.g. in terms of the chlorophyll concentration in Case 1 water. The EcoLight-S radiative transfer code (Mobley (2011)) was developed to provide extremely fast solutions of the radiative transfer equation to obtain $E_\text{d}(z)$ and $E_\text{u}(z)$ for any water body (deep or shallow, Case 1 or Case 2) for use in Eq. (likesection7) in coupled physical-biological-optical ecosystem models. (EcoLight-S also computes other quantities such as PAR and remote-sensing reflectance, which are inputs to photosynthesis calculations or are useful for model validation.) For an example of such a model, see Mobley et al. (2015).