In many cases we are interested only in the total radiance, without regard to its state of polarization. Most instruments used in optical oceanography are by design not sensitive to polarization. Furthermore, it is commonly assumed that many processes of interest, such as absorption of light by phytoplankton to drive photosynthesis, do not depend on the polarization state. (However, an internet search on "polarization dependent phytoplankton absorption" or similar key words results in exactly zero papers on this topic.) However, underwater light fields are partially linearly polarized by transmission through the sea surface and by scattering within the water column, even though the Sun's direct beam is unpolarized. Thus there will be a contribution to the total radiance through conversion of this linearly polarized light to unpolarized by the (1,2) element of the scattering matrix.

From the VRTE for Total Radiance to the SRTE

Now let $c = \cos 2\alpha$, $s' = \sin 2\alpha'$, etc. for the rotation angles seen in Eqs. (3) and (4) of the general VRTE page; let $M_{ij} = M_{i,j}(\psi)$; and use the form of $\mathbb{M}$ seen in Eq. (2) of the previous page. The phase matrix after matrix multiplications then becomes

$$
\mathbb{Z} = \begin{bmatrix}
M_{11} & c'\,M_{12} & -s'\,M_{12} & 0 \\
c\,M_{12} & c'\,c\,M_{22} - s'\,s\,M_{33} & -s'\,c\,M_{22} - c'\,s\,M_{33} & 0 \\
s\,M_{12} & c'\,s\,M_{22} + s'\,c\,M_{33} & -s'\,s\,M_{22} + c'\,c\,M_{33} & 0 \\
0 & 0 & 0 & M_{44}
\end{bmatrix}.
\tag{1}
$$

This equation shows that linear polarization (the Q and U Stokes parameters) contributes to the total I component via the M_{12} matrix element and the rotation angles that carry the incident linear polarization into the scattering plane. I does not depend on the rotation angle from the scattering plane to the final meridian place because total radiance is independent of the coordinate system. Scattering by the volume of the water body cannot convert circular polarization (V) into unpolarized light because $Z_{14} = 0$.

Writing Eq. (3) of the previous page for just the first element of the Stokes vector then gives

$$
\cos\theta \frac{d}{dz} I(z,\theta,\phi) = - c(z)\, I(z,\theta,\phi)
$$

$$
+ \iint_{4\pi} M_{11}(z,\psi)\, I(z,\theta',\phi')\, d\Omega(\theta',\phi') + Q_I(z,\theta,\phi)
$$

$$
+ \iint_{4\pi} \cos\alpha'\, M_{12}(z,\psi)\, Q(z,\theta',\phi')\, d\Omega(\theta',\phi')
$$

$$
- \iint_{4\pi} \sin\alpha'\, M_{12}(z,\psi)\, U(z,\theta',\phi')\, d\Omega(\theta',\phi').
\tag{2}
$$

Here Q_I is the first element of the source term $\mathbb{Q} = [Q_I, Q_Q, Q_U, Q_V]^T$, which contributes to the first element of the Stokes vector $\mathbb{S} = [I, Q, U, V]^T$.

Equation (likesection2) is the correct 1D radiative transfer equation for the total radiance $I(z,\theta,\phi)$, under the simplifications described above. Unfortunately, this equation cannot be solved as written because Q and U are not known unless the full VRTE, Eq. (3) of the previous page, is solved.

As already noted, routine solution of the VRTE is hindered not so much by the mathematics as by the lack of the needed inputs for the scattering matrices for various water-column constituents (phytoplankton, mineral particles, microbubbles, etc.). In response to this situation, modelers often raise their hands in surrender and simply drop the two integrals involving M_{12} in Eq. (likesection2). This amounts to an ad hoc assumption—almost always incorrect—that underwater radiance distributions are unpolarized. However, the result is a scalar RTE for the total radiance that requires only beam attenuation and the VSF as input. In particular, this is the SRTE that is solved by the HydroLight radiative transfer software. There are much more data and models available for the VSF than for the other elements of the scattering matrix, so the tradeoff is decreased accuracy in exchange for requiring less input and doing simpler mathematics (i.e., faster computer programs).

In optical oceanography it is common to write the total radiance as $L(z, \theta, \phi, \lambda)$ rather than $I((z, \theta, \phi, \lambda)$. The volume scattering function is usually written as either $\beta(z, \psi, \lambda) = M_{11}(z, \psi, \lambda)$, which emphasizes the scattering angle, or as $\beta(z; \theta', \phi' \to \theta, \phi; \lambda)$, which emphasizes the initial and final directions of the scattered light. With these changes in notation, the 1D SRTE as commonly seen in textbooks (e.g., Eq. (5.23) of Mobley (1994)) is

$$\cos\theta \frac{dL(z, \theta, \phi, \lambda)}{dz} = \& - c(z, \lambda) L(z, \theta, \phi, \lambda)$$

$$+ \& \int_0^{2\pi} \int_0^{\pi} L(z, \theta', \phi', \lambda) \beta(z; \theta', \phi' \to \theta, \phi; \lambda) \sin\theta' d\theta' d\phi'$$

$$+ \& S(z, \theta, \phi, \lambda) \ . \tag{3}$$

Following Light and Water (1994), the source term in the SRTE is denoted by S, which is not to be confused with the Stokes vector $\mathbb{S}$. This equation expresses location as geometric depth z and the IOPs in terms of the beam attenuation c and the volume scattering function β. This form is convenient for studies at multiple wavelengths because the geometric depth z is independent of the IOPs.

The beam attenuation coefficient is the sum of the absorption and scattering coefficients: $c = a + b$. The VSF is often written as the product of the scattering coefficient and the scattering phase function: $\beta(z, \psi, \lambda) = b(z, \lambda) \tilde{\beta}(\psi)$. Dividing the SRTE by c, defining the optical depth by $d\zeta = c \, dz$, and the albedo of single scattering by $\omega_o = b/c$, gives

$$\cos\theta \frac{dL(\zeta, \theta, \phi, \lambda)}{d\zeta} = \& - L(\zeta, \theta, \phi, \lambda)$$

$$+ \& \omega(\zeta, \lambda) \int_0^{2\pi} \int_0^{\pi} L(\zeta, \theta', \phi', \lambda) \tilde{\beta}(\zeta; \theta', \phi' \to \theta, \phi; \lambda) \sin\theta' d\theta' d\phi'$$

$$+ \& \frac{1}{c(\zeta, \lambda)} S(\zeta, \theta, \phi, \lambda) \ . \tag{4}$$

This equation expresses location as nondimensional optical depth ζ and the IOPs in terms of the albedo of single scattering and the scattering phase function. This form is convenient for theoretical studies at one wavelength. It is difficult to use for multiple wavelengths because

2

optical depth depends on the inherent optical properties. Thus the same geometric depth in meters corresponds to different optical depths at different wavelengths.

The magnitude of the error in computed values of the total radiance induced by ignoring the M_{12} terms in Eq. (likesection2) can be crudely estimated as follows. Measurements show that underwater light fields are partially linearly polarized (e.g., Cronin and Shashar (2001), Sabbah and Shashar (2006), You et al. (2011)). The degree of linear polarization is seldom more than 0.5, and is typically in the 0.1 to 0.3 range. Figure 4 of the previous page shows that the maximum magnitude of $M_{12}(\psi)/M_{11}(\psi)$ is about 0.8 at $\psi = 90$ deg. For $Q/I = 0.3$ and a 90 deg scattering angle, the ratio of $M_{12}Q$ (or $M_{12}U$) to the $M_{11}I$ term is then

$$\frac{M_{12}Q}{M_{11}I} = 0.8 \times 0.3 \approx 0.25.$$

Thus for a viewing direction at right angles to the unscattered direction (often the sun's direct beam in the water), single scattering gives an error of order 25%. However, for near-forward and near-backward scattering angles, $M_{12} \approx 0$, and the error is much smaller. In addition, multiple scattering tends to reduce the degree of polarization. It thus seems reasonable to say that neglecting the M_{12} terms in Eq. (likesection2) can cause errors in the total radiance as large as a few tens of percent in certain directions, but in other directions the errors may be of order 10% or even smaller. This error can be positive or negative, depending on the sign of Q and U. That is, in some directions the radiance value will be too large, and in other directions too small. These errors tend to cancel when computing irradiances, which are integrals of the radiance over direction. The historical justification for dropping the M_{12} terms in Eq. (likesection2) has been that, although the errors in the total computed radiances are of order 10% (and possibly greater), the errors in the corresponding irradiances are only of order 1%. That is considered an acceptable tradeoff for the increased simplicity of solving Eq. (likesection3) compared to the VRTE of the previous page.

Summary

This development started with QED and part-per-trillion accuracy and ended up with the SRTE and order of 10% accuracy in radiances (but perhaps several tens of percent error in particular directions), but only $\tilde{1}$% percent in irradiances. This loss of accuracy is the price to be paid for an equation that is simple enough to solve on a routine basis for situations of practical oceanographic interest.

The largest errors in computing underwater radiances usually come not from the approximations made to the general VRTE, but from the inputs to the simplified VRTE or SRTE. This is especially true if generic bio-geo-optical models for the absorption and scattering coefficients are used to convert chlorophyll and/or mineral particle concentrations to IOPs. Even if correct on average, such models can be off by tens of percent to an order of magnitude in any particular situation. A 10% error in the absorption coefficient can easily result in an order-of-magnitude error in the computed radiance or irradiance at depth. Guessing or modeling the phase function can give a factor-of-two or more error in backscatter directions, which gives an error of the same magnitude in the water-leaving radiance. It is certainly possible to get very good agreement between measurements and predictions based on the

SRTE, but heroic efforts are required to measure all of the needed inputs. Examples of such comparisons can be seen in Tzortziou et al. (2006) and Tonizzo et al. (2017).