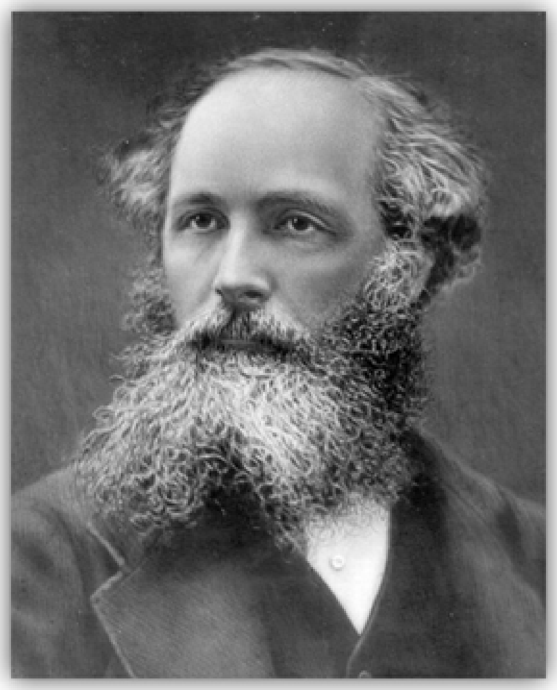

Figure 1: James Clerk Maxwell (1831-1879).
"War es ein Gott, der diese Zeichen schrieb?" ("Was it a God who wrote these symbols?") —Ludwig Boltzmann, commenting on Maxwell's equations (and recycling a quote from Goethe's Faust).

This page begins a qualitative overview of Maxwell's equations. Entire books have been written about these equations, so two pages are not going to teach you much. The goal here is to present the fundamental ideas and, hopefully, inspire you to continue to study these equations in the references provided. The discussion presumes a knowledge of basic physics (concepts such as electric charge and current, and electric and magnetic fields). Knowledge of vector calculus (divergence and curl in particular) is needed to understand the equations, but you can understand the basic ideas even without the math. If you are unfamiliar with the basic physics and math of electric and magnetic fields, or need a good review, an excellent place to start is A Student's Guide to Maxwell's Equations by Fleisch (2008). That tutorial spends 130 pages covering what is presented here.

Physical Preliminaries: Electric and Magnetic Fields

Recall the Lorentz equation for the force $\mathbf{F}$ exerted on an electric charge q moving with velocity $\mathbf{v}$ through an electric field $\mathbf{E}$ and a magnetic filed $\mathbf{B}$ (in SI units):

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

In this discussion, vectors in 3D space are indicated by bold-faced symbols. The $\times$ indicates the vector cross product. The Lorentz equation gives us the units for electric and magnetic fields. The force on the charge due to the electric filed is $\mathbf{F} = q\mathbf{E}$, so the units of electric field must be

$$[E] = \frac{[F]}{[q]} = \frac{\text{newton}}{\text{coulomb}} ,$$

where [...] denotes "units of ...". Similarly, magnetic fields have units of

$$[B] = \frac{[F]}{[qv]} = \frac{\text{newton}}{\text{coulomb meters per second}} .$$

You will see equivalent forms for these units. A newton per coulomb is the same as a volt per meter. An ampere is a current of a coulomb per second, so we can write $[B] = \mathrm{N}/(\mathrm{A\,m})$, which is called a Tesla (T). Table center1 summaries for reference the quantities seen in Maxwell's equations.

The first two quantities in Table center1 are worthy of comment. The electric constant or permittivity of free space, ϵ_o, is an empirical constant that measures an electric field's ability to "penetrate" a vacuum. In other words, it sets the strength of the force between two electric charges separated by some distance in a vacuum. This is seen if you write Coulomb's law as

$$F = \frac{1}{4\pi\epsilon_o} \frac{q_1\, q_2}{r^2} ,$$

where F is the magnitude of the force (in newtons) between charges q_1 and q_2 (in coulombs) separated by a distance r (in meters) in a vacuum. The value of ϵ_o is not derived from fundamental physics; it must be measured. This can be done by measuring the force between two charges, but is more accurately measured with a parallel plate capacitor. Similarly, the magnetic constant or permeability of free space, μ_o, measures a magnetic field's ability to penetrate a vacuum. It sets the strength of the magnetic force between two current-carrying wires separated by some distance in a vacuum. It also must be measured. Why do these two fundamental constants have the particular values shown in Table center1? This is a question like "why does an electron have the charge it has, and not some other value?" All that can be said is that these values are what they are because that is just how the universe works.

By the way, an electric field of 1 V/m is a very weak field: just think of a large parallel plate capacitor with the plates separated by 1 m and connected by a 1 V battery. The electric field between a thundercloud and the ground is of order 10^5 V/m just before a lightning discharge. On the other hand, a 1 T magnetic field is really strong. The Earth's magnetic field at the surface is about 5×10^{-5} T. Important research has shown that a 16 T magnetic field is so strong that it can overcome the force of gravity and levitate a living frog (Berry and Geim, 1997. *Eur. J. Phys* 18, 307-313).

Mathematical Preliminaries: Divergence and Curl

In order to enjoy Maxwell's equations, it is necessary to understand the mathematical notation. For the benefit of readers who are not familiar with vector calculus, the needed operations are as follows.

A *scalar field* $S(x, y, z, t)$ associates a number with each point in space and time. An example is the temperature in room. A *vector field* $\mathbf{V}(x, y, z, t) = \mathbf{V}(\mathbf{x}, t)$ associates a vector

Physical quantity	**Symbol**	**SI Units**	**Comment**
Electric constant	ϵ_o	$\approx 8.8542 \times 10^{-12} \mathrm{A}^2\,\mathrm{s}^4\,\mathrm{kg}^{-1}\,\mathrm{m}^{-3}$ (or $\mathrm{C}^2\,\mathrm{N}^{-1}\,\mathrm{m}^{-2}$)	measures a vacuum's ability to support an electric field; also called the permittivity of free space
Magnetic constant	μ_o	$\approx 1.2566 \times 10^{6} \mathrm{kg\,m\,s}^{-2}\,\mathrm{A}^{-2}$ (or $\mathrm{N\,A}^{-2}$)	measures a vacuum's ability to support a magnetic field; also called the permeability of free space
Electric charge	q	coulomb (C)	a fundamental physical quantity
Charge density	ρ	$\mathrm{C\,m}^{-3}$	charge per unit volume
Electric current	**I**	ampere (A = C/s)	measures flow of electric charge per unit time
Current density	**J**	$\mathrm{A\,m}^{-2}$	current per unit area

(a magnitude and a direction)

$$\mathbf{V}(x,y,z,t) = V_x(x,y,z,t)\hat{\mathbf{x}} + V_y(x,y,z,t)\hat{\mathbf{y}} + V_z(x,y,z,t)\hat{\mathbf{z}}$$

with each point in space and time. An example is the wind blowing outside your home.

The "del" operator ∇ (sometimes also called "nabla") can be thought of as a vector whose elements are partial derivative operators defined (in cartesian coordinates) as

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \, .$$

Applying the del operator to a scalar gives a vector, called the *gradient* of the scalar field:

$$\nabla S = \hat{\mathbf{x}}\frac{\partial S}{\partial x} + \hat{\mathbf{y}}\frac{\partial S}{\partial y} + \hat{\mathbf{z}}\frac{\partial S}{\partial z} \, .$$

Just like any vector, we can take the dot product of ∇ with a vector, and the result is a scalar. Taking the dot product of the del operator with a gradient gives a scalar:

$$\nabla \cdot \nabla S = \nabla^2 S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} \, .$$

This is usually called the *Laplacian* of S, and ∇^2 is called the Laplace operator.

The *divergence* of a vector field is defined as

$$\nabla \cdot \mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \, .$$

The cross product of two vectors $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\mathbf{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}}$ is

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\hat{\mathbf{x}} + (a_z b_x - a_x b_z)\hat{\mathbf{y}} + (a_x b_y - a_y b_x)\hat{\mathbf{z}} \, .$$

In the same fashion we get the *curl* of a vector field, which is the cross product of ∇ with the vector field and yields a vector:

$$\nabla \times \mathbf{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{\mathbf{z}} \, .$$

There is a useful trick for remembering the order of the vector components and derivatives in the curl if you know how to expand the determinant of a 3×3 matrix. Write the unit direction vectors in the first row of the determinant, the partial derivatives in the second row, and the vector components in the third row:

$$\nabla \times \mathbf{V} = \begin{vmatrix} \hat{\mathbf{x}}\&\hat{\mathbf{y}}\&\hat{\mathbf{z}} \\ \frac{\partial}{\partial x}\&\frac{\partial}{\partial y}\&\frac{\partial}{\partial z} \\ V_x\&V_y\&V_z \end{vmatrix} \, .$$

Then expand the determinant just as though the elements were ordinary numbers, and let the derivatives operate on the vector elements.

Thus the divergence and curl are just certain combinations of the spatial derivatives of a vector field. Each has a physical interpretation when the vector field is a physical variable such as the velocity or an electric field. However, just knowing the definitions is sufficient for our level of presentation of Maxwell's equations.

Maxwell's Equations in Vacuo

Without further ado, Maxwell's equations for the electric field $\mathbf{E}(\mathbf{x}, t)$ and magnetic field $\mathbf{B}(\mathbf{x}, t)$ in a vacuum are (in differential form, in SI units)

$$\nabla \cdot \mathbf{E} = \& \frac{1}{\epsilon_o}\rho \tag{1}$$

$$\nabla \cdot \mathbf{B} = \& 0 \tag{2}$$

$$\nabla \times \mathbf{E} = \& -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\nabla \times \mathbf{B} = \& \mu_o \mathbf{J} + \mu_o \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

Note that "in a vacuum" means that the electric and magnetic fields are in empty space. There can still be electric charges located here and there in space (the ρ term), and the same for currents ($\mathbf{J}$), which give rise to the fields in the region of interest.

These equations can be described as follows:

Eq.(likesection1) This equation is called Gauss's law for electric fields. It shows how electric charges (the charge density ρ) create electric fields. This equation is the equivalent of Coulomb's law for a point charge.

Eq.(likesection2) This equation is sometimes called Gauss's law for magnetic fields. It says that there are no magnetic charges corresponding to electric charges.

Eq.(likesection3) This is Faraday's law. It shows that a time-varying magnetic field creates an electric field.

Eq.(likesection4) This is Ampere's law as modified by Maxwell. The first term on the right, deduced by Ampere, shows that electric currents create magnetic fields. The second term on the right, added by Maxwell, shows that a time-varying electric field also creates a magnetic field.

Thus there are two ways to create electric fields: electric charges create them, and time-dependent magnetic fields create them. One might suppose that the electric fields resulting from these two entirely different creation mechanisms could some way be different, but they are not. An electric field is an electric field, no matter how it is created. That's just the way the universe works. (Pondering this equivalence of electric fields, no matter how created, was one of the things that lead Einstein to the development of special relativity.) The same situation holds for magnetic fields. They can be created by electric currents or by time-dependent electric fields, but the nature of the magnetic field is the same in either case.

Simply stating Maxwell's equations is really no different than simply stating Newton's law of gravity for the magnitude of the force of attraction between two spherical masses M_1 and M_2 separated by a distance r:

$$F = G\frac{M_1 M_2}{r^2}\,. \tag{5}$$

Newton did not *derive* his law of gravity from more fundamental principles; it *is* the fundamental principle. Newton found that if he *assumed* Eq. (likesection5) to be true, then he could derive Kepler's laws of planetary motion, the motion of the moon, and (to first order) the ocean tides. The same can be said of Maxwell's equations. They are based on decades of observational work by Coulomb, Gauss, Faraday, Ampere and others, but we can view them as the mathematical statement of the fundamental laws governing electric and magnetic fields. We can simply accept these equations as given and get on with the business of applying them to problems of interest. (Of course, "fundamental laws of nature" may turn out of be imperfect in the light of new data. That happened to Newton's law of gravity, which was replaced by, and can be derived from, Einstein's theory of general relativity. Likewise, Maxwell's equations can now be derived from the more fundamental laws of quantum electrodynamics developed by Feynman and others.)

It may at first glance seem that Maxwell's equations are over-determined. That is, there are four equations but only two unknowns, $\mathbf{E}$ and $\mathbf{B}$. This would be true for algebraic equations, in which case we could solve two linearly independent equations for two unknowns. However, for vector fields, Helmholtz's theorem (also known as "the fundamental theorem of vector calculus") says that an arbitrary vector field in 3 dimensions can be uniquely decomposed into a divergence part (with zero curl) and a curl part (with zero divergence) (under a few conditions, namely vector functions that are sufficiently smooth and that decay to zero at infinity). Conversely, knowing the divergence and curl of a vector field determines the vector field. That is the case here for both $\mathbf{E}$ and $\mathbf{B}$. Given the charge density ρ and current density $\mathbf{J}$, the four Maxwell equations uniquely determine the electric and magnetic fields via their divergences and curls. (To be rigorous, a vector field is determined from its divergence and curl to within an additive term. This is somewhat like saying that knowing a derivative $df(x)/dx$ determines f to within an additive constant. Adding a boundary condition $f(x_o) = f_o$ then fixes the value of the constant.)

Light as an Electromagnetic Phenomenon

Starting with equations (likesection1) to (likesection4), Maxwell derived what is probably the most elegant and important result in the history of physics. Consider a region of space where there are no charges ($\rho = 0$) or currents ($\mathbf{J} = 0$). Equations (likesection1)-(likesection4) then become

$$\nabla \cdot \mathbf{E} = \&0 \tag{6}$$

$$\nabla \cdot \mathbf{B} = \&0 \tag{7}$$

$$\nabla \times \mathbf{E} = \& -\frac{\partial \mathbf{B}}{\partial t} \tag{8}$$

$$\nabla \times \mathbf{B} = \& \mu_o \epsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{9}$$

Now take the curl of Eq. (likesection8), use the vector calculus identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$, use Eq. (likesection6) to eliminate the $\nabla(\nabla \cdot \mathbf{E})$ term, and use Eq. (likesection9) to rewrite the $\partial(\nabla \times \mathbf{B})/\partial t$ term. The result is

$$\nabla^2 \mathbf{E} = \mu_o \epsilon_o \frac{\partial^2 \mathbf{E}}{\partial t^2} \,.$$

The same process starting with the curl of Eq. (likesection9) gives an equation of the same form for $\mathbf{B}$. Equations of the form

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

describe a wave propagating with speed v. Thus each component of $\mathbf{E}$ and $\mathbf{B}$ satisfies a wave equation with a speed of propagation

$$v = \frac{1}{\sqrt{\mu_o \epsilon_o}} \,. \tag{10}$$

Inserting the experimentally determined values of μ_o and ϵ_o given in Table center1 gives $v = 3 \times 10^8 \mathrm{m\,s^{-1}}$. As Maxwell observed (in *A Dynamical Theory of the Electromagnetic Field*, 1864, §20), "This velocity is so nearly that of light that it seems we have strong reason to conclude that light itself (including radiant heat and other radiations) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws." *This conclusion is one of the greatest intellectual achievements of all time: not only were electric and magnetic fields tied together in Maxwell's equations, but light itself was shown to be an electromagnetic phenomenon.* This is the first example of a "unified field theory," in which seeming independent phenomena—here electric fields, magnetic fields, and light—were shown to related and governed by the same underlying equations.